

NILPOTENCY AND H -SPACES

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§1. INTRODUCTION

It is well-known that if L is an H -space then ΩL is homotopy-abelian or, in the notation of Bernstein–Ganea [1],

$$\text{nil } L = 1;$$

here $\text{nil } L$ refers to the nilpotency of ΩL as a group in the category \mathcal{T}_h of based spaces and based homotopy classes. The converse implication, that $\text{nil } L = 1$ implies that L is an H -space, holds under certain strong restrictions[‡] on the homotopy groups of L ; that it is false in general was demonstrated first by Bernstein–Ganea, taking the example of a 2-stage Postnikov system

$$K(Z, 5) \rightarrow L_1 \rightarrow K(Z, 2)$$

in which the k -invariant is the cube of the fundamental class. Then $\Omega L_1 \simeq S^1 \times K(Z, 4)$ and carries the multiplication of $S^1 \times K(Z, 4)$ which is certainly commutative. On the other hand if one takes the 2-stage Postnikov system

$$K(Z, 3) \rightarrow L_2 \rightarrow K(Z, 2)$$

in which the k -invariant is the square of the fundamental class then $\Omega L_2 \simeq S^1 \times K(Z, 2)$, but the multiplication on ΩL_2 is not the obvious multiplication on $S^1 \times K(Z, 2)$, that is, its multiplication as $\Omega(K(Z, 2) \times K(Z, 3))$. This may be seen by looking at the Pontryagin ring $H_*(\Omega L_2)$. By a standard spectral sequence argument (or by looking at the Postnikov decomposition of S^2) one shows that if a generates $H_1(\Omega L_2)$ and b is the image in $H_2(\Omega L_2)$ of the generator of $H_2(K(Z, 2))$, then $a^2 = b$ (whereas, of course, $a^2 = 0$ in $H_*(\Omega(K(Z, 2) \times K(Z, 3)))$). In fact the multiplication on ΩL_2 is not commutative; for the calculation of $H_*(\Omega L_2)$ shows that the multiplication on ΩL_2 , which, when transferred to the homotopically equivalent space $S^1 \times K(Z, 2)$ may be regarded as determining a pair of elements

$$\mu_i \in H^i(S^1 \times S^1 \times K(Z, 2) \times K(Z, 2)), \quad i = 1, 2,$$

is given by $\mu_1 = \alpha_1 + \alpha_2$, $\mu_2 = \alpha_1\alpha_2 + \beta_1 + \beta_2$, where α_i generates $H^1(S^1)$, S^1 being

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[‡] See [11].

embedded as the i th copy, and β_i generates $H^2(K(Z,2))$, $K(Z,2)$ being embedded as the i th copy. Since $\alpha_1\alpha_2 = -\alpha_2\alpha_1$, the multiplication is plainly non-commutative†.

The spaces L of these two examples are special cases of spaces F obtained as fibres of maps $f: X \rightarrow Y$ such that $\Omega f \simeq 0$. In any such case we have a homotopy equivalence

$$(1.1) \quad \Omega F \simeq \Omega(X \times \Omega Y),$$

but it is not necessarily the case, as we have seen, that the homotopy equivalence (1.1) can be given by an H -map, that is, a homotopy-homomorphism. In the first example it could be so given—and could only be so given—because any space of the homotopy type of $S^1 \times K(Z,4)$ can carry only one H -structure (up to homotopy); in the second example we have seen that it cannot be so given. We look here for less special reasons why it should be possible to realize (1.1) by an H -map (so that we have a H -homotopy-equivalence, or H -equivalence (1.1)). In fact we look at the kernel of the “cohomology suspension” $\Omega_*: \Pi(X, Y) \rightarrow \Pi(\Omega X, \Omega Y)$ and study the subset of this kernel consisting of those classes $\{f\}$ for which (1.1) is an H -equivalence; the members of this subset are called P -classes. We note that if (1.1) is an H -equivalence and X is not contractible then

$$(1.2) \quad \text{nil } F = \text{nil } X,$$

since $\text{nil}(A \times B) = \max(\text{nil } A, \text{nil } B)$.

In §2 we obtain the general algebraic properties of the set of P -classes; and we also prove that there does, in fact, exist a substantial family of non-trivial P -classes. By taking Y to be an Eilenberg–MacLane space we obtain cohomology P -classes and our general theorems on the algebraic nature of such classes show, in particular, why the class in $H^6(K(Z,2))$ used in the first example is a P -class (without appealing to special dimensionality features present in this special case which obscure the general nature of the concept).

Now in the special case when X is an H -space (and f belongs to a P -class), (1.2) shows that $\text{nil } F = 1$; on the other hand we show, by computing higher Whitehead products in F in §3, that the spaces F are not in general H -spaces even when X is an H -space, so that our construction yields a rich source of spaces F which are not H -spaces but whose loop-spaces are homotopy-abelian. Certainly in the first example L_1 is such a space; here the argument is simply that, for 2-stage Postnikov systems, the space L can only be an H -space if the k -invariant is primitive and the generator of $H^6(K(Z,2))$ is certainly not primitive. In fact, by applying standard homotopy theory to the example covered in (3.7), where a non-trivial secondary Whitehead product occurs, one may show that, in the first example,

$$\pi_2(L_1) = Z = \langle \alpha \rangle, \quad \pi_5(L_1) = Z = \langle \beta \rangle, \quad \text{and} \quad [\alpha, \alpha, \alpha] = 6\beta,$$

so that L_1 does have non-trivial secondary Whitehead products.

We conclude §3 by pointing out that, although F is not itself in general an H -space, the function space F^K is an H -space, provided K is a compact Hausdorff space with

$$\text{w cat } K < m,$$

† This could also be seen from the Pontryagin ring, since if it were commutative we would have $2b = 0$.

where $w \operatorname{cat} K$ is the numerical invariant introduced in [2] and m is positive integer depending on the construction of F ; in the case of the space L_1 we would have $m = 3$ and we could, for example, infer that L_1^K is an H -space provided K is connected and its Lusternik–Schnirelmann category did not exceed 3.

Section 4 studies, somewhat rapidly, the situation dual to that considered theretofore. However, the paper leaves open many problems, conspicuously that of giving a complete description of the cohomology P -classes for given spaces X .

A summary version of this paper appeared in the Proceedings of the Aarhus Colloquium on Algebraic Topology [7]. The author wishes to express his warm gratitude to T. Ganea for many very helpful and stimulating discussions during the early stage of the work. In particular, many of the results of §3 (in particular Theorem (3.8) and Corollary (3.14)) were conjectured—and even proved—by him.

The notations used here, insofar as they involve category-theoretic notions, are those of [3], with the exception that, wishing to reserve the symbol $\{ \}$ to indicate the homotopy class of a map, we have used (f_1, f_2) for the map into a (direct) product whose components are f_1 and f_2 . The category \mathcal{T} is, as usual, the category of based spaces of the based homotopy type of countable CW -complexes and based maps.

§II. THE P -PROPERTY FOR MAPS

We wish to study a particular family of maps $f: X \rightarrow Y$ in the category \mathcal{T} . Let $j: F \rightarrow X$ be the fibre of f and let $s: \Omega Y \rightarrow F$ be the fibre of j . There is thus an “exact sequence”, determined by f ,

$$(2.1) \quad \dots \rightarrow \Omega^2 Y \xrightarrow{\Omega s} \Omega F \xrightarrow{\Omega j} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{s} F \xrightarrow{j} X \xrightarrow{f} Y.$$

For any $A \in \mathcal{T}$, the functor $\Pi(A, \quad)$ maps (2.1) to an exact sequence and, as is well known $(\Omega s)_* \Pi(A, \Omega^2 Y)$ is in the centre of $\Pi(A, \Omega F)$. We wish to study those maps f such that $\Omega f \simeq 0$. In this case we get a short exact sequence of group-homomorphisms

$$(2.2) \quad 0 \rightarrow \prod(A, \Omega^2 Y) \xrightarrow{(\Omega s)_*} \prod(A, \Omega F) \xrightarrow{(\Omega j)_*} \prod(A, \Omega X) \rightarrow 0.$$

Moreover there is a map $m: \Omega X \rightarrow \Omega F$ such that $(\Omega j)m \simeq 1$, so that $(\Omega j)_* m_* = 1$, and we may ask when m may be taken to be an H -map (so that m_* is a homomorphism). We first prove an algebraic lemma.

LEMMA (2.3). Let $0 \rightarrow A \xrightarrow{\lambda} B \xrightarrow{\mu} C \rightarrow 0$ be an exact sequence of groups in which λA is in the centre of B . Then the following 3 assertions are equivalent:†

- (i) $\exists \bar{\mu}: C \rightarrow B$ with $\mu \bar{\mu} = 1$;
- (ii) $\exists \bar{\lambda}: B \rightarrow A$ with $\bar{\lambda} \lambda = 1$;
- (iii) $\exists \bar{\lambda}: B \rightarrow A$ such that $(\bar{\lambda}, \mu): B \cong A \times C$.

Proof. We permit ourselves to suppress the symbol λ from the notation, regarding A as a central subgroup of B .

† Of course, the maps $\bar{\mu}$, $\bar{\lambda}$ postulated in (i), (ii), (iii) are homomorphisms.

(i) \Rightarrow (ii). Given $\bar{\mu}$, set $\bar{\lambda}b = b(\bar{\mu}\mu b)^{-1}$. Then $\bar{\lambda}b \in A$ because $\mu\bar{\mu} = 1$, and

$$\begin{aligned}\bar{\lambda}(b)\bar{\lambda}(b') &= b(\bar{\mu}\mu b)^{-1}b'(\bar{\mu}\mu b')^{-1} \\ &= bb'(\bar{\mu}\mu b')^{-1}(\bar{\mu}\mu b)^{-1}, \text{ because } A \text{ is in the centre of } B \\ &= bb'(\bar{\mu}\mu(bb'))^{-1} \\ &= \bar{\lambda}(bb'),\end{aligned}$$

so that $\bar{\lambda}$ is a homomorphism. Moreover, if $a \in A$, $\bar{\lambda}a = a$ since $\mu a = e$. (ii) \Rightarrow (iii). We take the homomorphism $\bar{\lambda}$ satisfying (ii). Then, first, $(\bar{\lambda}, \mu)$ is onto. For given $a \in A$, $c \in C$, let $\mu b_0 = c$ and set $b = b_0(\bar{\lambda}b_0)^{-1}a$. Plainly $\mu b = \mu b_0 = c$ and $\bar{\lambda}b = a$ since $\bar{\lambda}$ is a retraction of B onto A . Second, $(\bar{\lambda}, \mu)$ is mono. For if $\bar{\lambda}b = e$, $\mu b = e$, then $b \in A$ and $e = \bar{\lambda}b = b$.

(iii) \Rightarrow (i). Given $\bar{\lambda}$ satisfying (iii), let θ be inverse to $(\bar{\lambda}, \mu)$ and set $\bar{\mu}c = \theta(e, c)$, $c \in C$. Then

$$(e, c) = (\bar{\lambda}, \mu)\theta(e, c) = (\bar{\lambda}, \mu)\bar{\mu}c = (\bar{\lambda}\bar{\mu}c, \mu\bar{\mu}c),$$

so that $\mu\bar{\mu} = 1$ (and $\bar{\lambda}\bar{\mu} = 0$).

Thus we may unambiguously describe the sequence of Lemma (2.3) as *splitting* if it satisfies any of the conditions (i), (ii), (iii). We now return to (2.1) and prove

THEOREM (2.4). *The following 4 assertions about the map $f: X \rightarrow Y$ are equivalent:*

- (a) Ωj has a right H -homotopy-inverse;
- (b) Ωs has a left H -homotopy-inverse (and $\Omega f \simeq 0$);
- (c) there is an H -map $k: \Omega F \rightarrow \Omega^2 Y$ such that the map $(\Omega j, k): \Omega F \rightarrow \Omega X \times \Omega^2 Y$ is a homotopy equivalence;
- (d) for each $A \in \mathcal{T}$, there is a splitting exact sequence (2.2) and the splitting is natural with respect to A .

Proof. (a) \Rightarrow (d). It is plain, by the exactness of (2.1) that, given (a), there is a short exact sequence (2.2) in which Ωj_* has a right inverse which is natural with respect to A .

(d) \Rightarrow (a). We are given a family of splitting homomorphisms

$$\bar{\mu}_A: \prod(A, \Omega X) \rightarrow \prod(A, \Omega F), (\Omega j_*)\bar{\mu}_A = 1,$$

which is natural with respect to A . Let $m: \Omega X \rightarrow \Omega F$ belong to the homotopy class $\bar{\mu}_{\Omega X}(1)$. It is then easy to see that $\bar{\mu}_A = m_*$. It further follows that m is an H -map since m_* is always a homomorphism (converse of Theorem (4.7) of [3]). Also

$$\Omega j_*\{m\} = \Omega j_*\bar{\mu}_{\Omega X}(1) = 1,$$

so that $(\Omega j)m \simeq 1$ and (a) is proved.

(b) \Rightarrow (d). The proof here is almost exactly as above. (One should observe that it follows easily from the proof of Lemma (2.3) that Ωj_* has a *natural* right inverse if and only if Ωs_* has a *natural* left inverse.)

(d) \Rightarrow (c). By the same reasoning as above we know that the natural homomorphism $\bar{\lambda}_A: \prod(A, \Omega F) \rightarrow \prod(A, \Omega^2 Y)$, left inverse to Ωs_* , is induced by an H -map $k: \Omega F \rightarrow \Omega^2 Y$. Moreover, by Lemma (2.3),

$$(\Omega j, k)_*: \prod(A, \Omega F) \cong \prod(A, \Omega X \times \Omega^2 Y), \quad \text{all } A \in \mathcal{T}.$$

It is now a standard deduction that $(\Omega j, k)$ is a homotopy equivalence and (c) is proved.

(c) \Rightarrow (a). We imitate the last part of the proof of Lemma (2.3). Let $w: \Omega X \times \Omega^2 Y \rightarrow \Omega F$ be homotopy inverse to $(\Omega j, k)$ and let $m: \Omega X \rightarrow \Omega F$ be given by $m(\omega) = w(\omega, *)$, $\omega \in \Omega X$. Then m is an H -map and $m = wi$, where i embeds ΩX in $\Omega X \times \Omega^2 Y$. Thus

$$i \simeq (\Omega j, k)wi = (\Omega j, k)m = ((\Omega j)m, km),$$

whence $(\Omega j)m \simeq 1$ (and $km \simeq 0$). This proves (a) and completes the proof of the theorem.

DEFINITION (2.5). *We say that the map $f: X \rightarrow Y$ has property P or is a P -map if it has any of the equivalent properties (a)–(d) of Theorem (2.4). If f is a P -map we may call F a P -fibre of X .*

The set of P -maps $X \rightarrow Y$ is, of course, a subset of those maps f such that $\Omega f \simeq 0$. In general it is a proper subset as the example given in the Introduction shows. It plainly contains the constant map since then $F = X \times \Omega Y$; before establishing that it is in general a non-trivial subset we first obtain some elementary properties of P -maps.

PROPOSITION (2.6). *If f is a P -map so are fu , vf for all (appropriate) u, v .*

Proof. Consider the diagram

$$\begin{array}{ccccccc} & & s' & & j' & & f' \\ \Omega Y & \xrightarrow{\quad} & F' & \xrightarrow{\quad} & X' & \xrightarrow{\quad} & Y \\ \downarrow 1 & & \downarrow s & & \downarrow j & & \downarrow u \\ \Omega Y & \xrightarrow{\quad} & F & \xrightarrow{\quad} & X & \xrightarrow{\quad} & Y \\ & & & & f & & \downarrow 1 \end{array}$$

where $f' = fu$ and j' is the fibre of f' . Then there is a map $t: F' \rightarrow F$ with $jt = uj'$, $ts' = s$. Thus $(\Omega t)(\Omega s') = \Omega s$. Since f is a P -map, Ωs has a left H -homotopy-inverse $k: \Omega F \rightarrow \Omega^2 Y$ (condition (b)). But then $k(\Omega t)(\Omega s') = k(\Omega s) \simeq 1$, so that $\Omega s'$ has a left H -homotopy-inverse $k(\Omega t)$ and f' is a P -map (obviously $\Omega f' \simeq 0$).

Next consider the diagram

$$\begin{array}{ccccc} & & j & & f \\ F & \xrightarrow{\quad} & X & \xrightarrow{\quad} & Y \\ & & \downarrow 1 & & \downarrow v \\ & & j' & & f' \\ F' & \xrightarrow{\quad} & X & \xrightarrow{\quad} & Y' \end{array}$$

where $f' = vf$ and j' is the fibre of f' . Then there is a map $t: F \rightarrow F'$ with $j't = j$. Since f is a P -map, Ωj has a right H -homotopy-inverse $m: \Omega X \rightarrow \Omega F$ (condition (a)). But then $(\Omega j')(\Omega t)m = (\Omega j)m \simeq 1$, so that $\Omega j'$ has a right H -homotopy-inverse $(\Omega t)m$ and f' is a P -map.

PROPOSITION (2.7). *If f_1, f_2 are P -maps so is $f_1 \times f_2$.*

Proof. Obvious.

PROPOSITION (2.8). *Let $f: X \rightarrow Y$, $g: X \rightarrow Z$ and let $c: Y \times Z \rightarrow C$ be the identification map which pinches Z , embedded in $Y \times Z$, to a point. Then if f is a P -map, so is*

$$h = c \circ (f, g): X \rightarrow C.$$

Proof. Let $l: L \rightarrow X$ be the fibre of h ,

$$L \xrightarrow{\quad l \quad} X \xrightarrow{\quad h \quad} C,$$

and, as usual, let $j: F \rightarrow X$ be the fibre of f .

$$F \xrightarrow{j} X \xrightarrow{f} Y.$$

Now $fj \simeq 0$, so $(f, g)j = (fj, gj) \simeq (0, gj)$, and $hj \simeq c(0, gj) = 0$. It follows that there is a map $t: F \rightarrow L$ with $lt \simeq j$. If f is a P -map, there is an H -map $m: \Omega X \rightarrow \Omega F$ with $(\Omega j)m \simeq 1$ (condition (b)). Then $(\Omega l)(\Omega t)m \simeq (\Omega j)m \simeq 1$, so that Ωl has a right H -homotopy-inverse, $(\Omega t)m$, and h is a P -map.

COROLLARY (2.9). *Let $q: Y \times Z \rightarrow Y \wedge Z$ be the projection onto the smashed product. Then if f is a P -map, so is $q^\circ(f, g)$.*

For q factors through c ; we apply Proposition (2.6).

PROPOSITION (2.10). *Property P is a property of homotopy classes.*

Proof. This is almost self-evident, and may in any case be deduced from the homotopy invariance of (2.2) and condition (d) for P -maps. A direct proof is, however, available as follows. It is plainly sufficient (in the light of Proposition (2.6)) to show that if $B: X \times I \rightarrow Y$ is a homotopy of f and if f is a P -map so is B . Let i embed X in $X \times I$ as $X \times 0$ and let $p: X \times I \rightarrow X$ be the projection. We have the diagram

$$\begin{array}{ccccc} F & \xrightarrow{j} & X & \xrightarrow{f} & Y \\ \downarrow t & & \downarrow i & & \downarrow 1 \\ \Phi & \xrightarrow{J} & X \times I & \xrightarrow{B} & Y, \end{array}$$

where J is the fibre of B . If $m: \Omega X \rightarrow \Omega F$ is an H -map with $(\Omega j)m \simeq 1$, then set $M = (\Omega t)m(\Omega p): \Omega(X \times I) \rightarrow \Omega\Phi$. Thus M is an H -map and $(\Omega J)M = (\Omega J)(\Omega t)m(\Omega p) = (\Omega i)(\Omega j)m(\Omega p) \simeq (\Omega i)(\Omega p) \simeq 1$. So B is a P -map and the proposition follows.

Let us consider the cases when Y is an Eilenberg–MacLane space. We suppose X connected and invoke the natural isomorphism between the reduced cohomology group $H^n(X; G)$ and $\Pi(X, K(G, n))$ to transfer property P from homotopy classes of maps to cohomology classes. We may thus speak of P -classes in the cohomology of X . Of course, all such classes are in the kernel of the cohomology suspension. We prove

THEOREM (2.11). (i) *The P -classes in $H^n(X; G)$ form a subgroup. More generally, if $u_i \in H^{n_i}(X; G_i)$, $i = 1, \dots, m$, are P -classes and T is an m -ary cohomology operation then $T(u_1, \dots, u_m)$ is a P -class.*

(ii) *The cup-product of two cohomology classes is a P -class if at least one of the factors is a P -class. More generally, if $u_i \in H^{n_i}(X; G_i)$, $i = 1, \dots, m$, and T is a normal m -ary cohomology operation, then $T(u_1, \dots, u_m)$ is a P -class if at least one of u_1, \dots, u_m is a P -class.*

(iii) *If $g: X_2 \rightarrow X_1$, then $g^*: H^*(X_1; G) \rightarrow H^*(X_2; G)$ maps P -classes to P -classes.*

Proof. (i) We prove the general statement. Let u_i be represented by $f_i: X \rightarrow K(G_i, n_i)$. Then $T(u_1, \dots, u_m)$ is represented by a map

$$X \xrightarrow{f_1 \times \dots \times f_m} K(G_1, n_1) \times \dots \times K(G_m, n_m) \xrightarrow{T} K(G, n),$$

where d is the diagonal map and t is the defining map of the operation T . It follows from Propositions (2.7) and (2.6) that this composite map is a P -map if each f_i is a P -map.

(ii) We prove the general statement. The operation T is normal if and only if t factors through $q: K(G_1, n_1) \times \dots \times K(G_m, n_m) \rightarrow K(G_1, n_1) \wedge \dots \wedge K(G_m, n_m)$, where q projects onto the smashed product. Thus (in the light of Proposition (2.6)) it suffices to show that

$$X \xrightarrow{(f_1, \dots, f_m)} Y_1 \times \dots \times Y_m \xrightarrow{q} Y_1 \wedge \dots \wedge Y_m$$

is a P -map if one of f_1, \dots, f_m is a P -map. Now there is no real loss of generality in supposing that f_1 is a P -map. Then $q \circ (f_1, \dots, f_m)$ plainly factors through

$$X \xrightarrow{(f_1, (f_2, \dots, f_m))} Y_1 \times (Y_2 \times \dots \times Y_m) \xrightarrow{q_1} Y_1 \wedge (Y_2 \times \dots \times Y_m),$$

where q_1 is the projection; and this latter map is a P -map by Corollary (2.9).

(iii) This follows immediately from Proposition (2.6).

Remark (2.12). (a) To summarize part of Theorem (2.11), we may say that the cohomology P -classes form an ideal which is closed under cohomology operations; and the association of the set of such classes with a space X is functorial.

(b) Theorem (2.11) remains valid for any cohomology theory given by maps into a space (e.g., K -theory and KO -theory). By generalizing the discussion to spectra—a mechanical operation—it remains valid for any cohomology theory.

(c) The reader should compare assertion (ii) with Corollary (2.15).

There remains the question of establishing the existence of non-trivial P -maps (we know that any nullhomotopic map is a P -map). In fact we will demonstrate the existence of a substantial class of P -maps.

Let $P_m = X_1 \times \dots \times X_m$ be the product of m spaces X_1, \dots, X_m and let $T_n \subseteq P_m$, $n = 0, 1, \dots, m$, be the union of the products of the X_i n at a time. Form the cofibration sequence

$$(2.13) \quad T_n \xrightarrow{k} P_m \xrightarrow{q} Q_n,$$

where k embeds T_n in P_m and q projects P_m onto the quotient of P_m by T_n .

THEOREM (2.14). *The map $q: P_m \rightarrow Q_n$ is a P -map if $n \geq 2$.*

Proof. Let $p_i: P_m \rightarrow X_i$, $u_i: X_i \rightarrow T_n$ be projections and injections. It is then plain, and standard, that the map $\dagger l = \Omega(u_1 p_1) + \dots + \Omega(u_m p_m)$ is a right homotopy inverse of Ωk ,

$$(\Omega k)l \simeq 1: \Omega P_m \rightarrow \Omega P_m.$$

We show that l is an H -map if $n \geq 2$. According to [4] it suffices to show that $\Omega(u_i p_{i_1})$ and $\Omega(u_i p_{i_2})$ strongly commute for any pair of distinct indices i_1, i_2 ; we may suppose $i_1 < i_2$.

Let e_1, e_2 embed X_{i_1}, X_{i_2} in $X_{i_1} \times X_{i_2}$ and let u embed $X_{i_1} \times X_{i_2}$ in T_n ; here we use the condition $n \geq 2$. Then $\Omega e_1, \Omega e_2$ strongly commute [4] and

$$u_{ij} p_{ij} = u e_j p_{ij}, j = 1, 2.$$

\dagger We need not specify the bracketing of the sum yielding l .

It thus follows from [4] that $\Omega(u_{i_1}p_{i_1})$ and $\Omega(u_{i_2}p_{i_2})$ strongly commute so that l is an H -map.

Now let $j: F \rightarrow P_m$ be the fibre of q . Then k factors as jr , where $r: T_n \rightarrow F$ and so $(\Omega j)(\Omega r)l = (\Omega k)l \simeq 1$. Thus Ωj has a right H -homotopy-inverse $(\Omega r)l$ and so q is a P -map by condition (a).

COROLLARY (2.15). *Let $u_i \in H^{n_i}(X, G_i)$, $i = 1, 2, \dots, m$, and let T be a normal m -ary cohomology operation; then $T(u_1, \dots, u_m)$ is a P -class provided $m \geq 3$.*

Proof. If u_i is represented by $f_i: X \rightarrow K(G_i, n_i)$, then $T(u_1, \dots, u_m)$ is represented by

$$X \xrightarrow{(f_1, \dots, f_m)} K(G_1, n_1) \times \dots \times K(G_m, n_m) \xrightarrow{q} K(G_1, n_1) \wedge \dots \wedge K(G_m, n_m) \xrightarrow{t} K(G, k).$$

But q is a P -map, by Theorem (2.14), since $m \geq 3$, so that $T(u_1, \dots, u_m)$ is a P -class by Proposition (2.6).

Remark (2.16). (a) Corollary (2.15) reduces the interest of Theorem (2.11) (ii) to the case $m = 2$.

(b) The example in the Introduction shows that Theorem (2.14) is false for $n = 1$ and Corollary (2.15) is false for $m = 2$. Indeed the map $l: \Omega P_m \rightarrow \Omega T_n$ is not an H -map in general if $n = 1$. For if it were the homomorphism $k_*: \pi_1(T_1) \rightarrow \pi_1(P_m)$ would have a right inverse; but this is the homomorphism from the free product of the groups $\pi_1(X_i)$, $i = 1, \dots, m$, to their direct product, and this certainly does not admit a right inverse in general.

(c) It follows from Corollary (2.15) that if u is a cohomology class in $H^k(X; G)$ which is a sum of triple cup-products, then u is a P -class. If F is the corresponding P -fibre, that is, if F is obtained from X by killing u , then

$$(2.17) \quad \Omega F \xrightarrow{H} \Omega X \times K(G; k-2),$$

that is, ΩF is H -homotopy-equivalent to $\Omega X \times K(G; k-2)$; in particular, the Pontryagin rings of ΩF and $\Omega X \times K(G; k-2)$ are isomorphic. Thus the Pontryagin ring of ΩF fails to distinguish between u and the zero class in $H^k(X; G)$.

Of course (2.17) is a special case of condition (c) for P -maps: if $f: X \rightarrow Y$ is a P -map and F is the P -fibre, then

$$(2.18) \quad \Omega F \xrightarrow{H} \Omega X \times \Omega^2 Y.$$

Thus, in particular, if X is non-contractible and F is a P -fibre of X we have

$$(2.19) \quad \text{nil } F = \text{nil } X.$$

§III. SECONDARY WHITEHEAD PRODUCTS

We revert to (2.10) but now confine attention to the case $n = m - 1$ and suppress the suffix $m - 1$ from the notation,

$$(3.1) \quad T \xrightarrow{k} P_m \xrightarrow{q} Q, \quad m \geq 3.$$

Let $g_i: S^{n_i} \rightarrow X_i$ represent $\alpha_i \in \pi_{n_i}(X_i)$, $i = 1, \dots, m$, $n_i > 1$, and let $N = n_1 + \dots + n_m$. The maps g_i induce in an obvious way a commutative diagram

$$(3.2) \quad \begin{array}{ccccc} \bar{T} & \xrightarrow{\bar{k}} & \bar{P}_m & \xrightarrow{\bar{q}} & \bar{Q} \\ \downarrow g_T & & \downarrow g_P & & \downarrow g_Q \\ T & \xrightarrow{k} & P_m & \xrightarrow{q} & Q \end{array},$$

where the top row represents the cofibration (3.1) relating to the spaces S^{n_1}, \dots, S^{n_m} . Note that $\bar{Q} = S^N$. Let $h: Q \rightarrow Y$ be a map such that $\dagger (hg_Q)_* \pi_N(\bar{Q}) \neq 0$ and let $j: F \rightarrow P_m$ be the fibre of $f = hq: P_m \rightarrow Y$. Let η generate the cyclic infinite subgroup $\partial \pi_N(\bar{P}_m, \bar{T})$ of $\pi_{N-1}(\bar{T})$. Since $qk = 0$, the map $k: T \rightarrow P_m$ factors through j as $k = jc$, $c: T \rightarrow F$. Set $g = cg_T: \bar{T} \rightarrow F$. We prove

LEMMA (3.3). $g_*(\eta) \neq 0$.

Proof. Consider the diagram

$$\begin{array}{ccccc} \pi_{N-1}(\bar{T}) & \xleftarrow{\partial} & \pi_N(\bar{P}_m, \bar{T}) & \xrightarrow{\bar{q}_*} & \pi_N(\bar{Q}) \\ \downarrow g_{T*} & & \downarrow g_{P*} & & \downarrow g_{Q*} \\ \pi_{N-1}(T) & \xleftarrow{\partial} & \pi_N(P_m, T) & \xrightarrow{q_*} & \pi_N(Q) \\ \downarrow c_* & & \downarrow c_* & & \downarrow h_* \\ \pi_{N-1}(F) & \xleftarrow{\partial} & \pi_N(j) & \xrightarrow{f_*} & \pi_N(Y) \end{array}$$

The commutativity is evident except perhaps for the bottom right-hand square. Recall that $j: F \rightarrow P_m$ is, in fact, the fibration induced by f from the canonical fibration $p: EY \rightarrow Y$. Thus $F \subseteq P_m \times EY$ consists of pairs (x, l) with $fx = l(0)$. The map $c: T \rightarrow F$ is given by $c(x) = (x, *)$, $x \in T$. Also we have the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{u} & EY \\ \downarrow j & & \downarrow p \\ P_m & \xrightarrow{f} & Y \end{array}$$

where $u(x, l) = l$, and $f_*: \pi_N(j) \rightarrow \pi_N(Y)$ is just $(u, f)_*$, followed by the canonical identification of $\pi_N(p)$ with $\pi_N(Y)$. (Of course, $f_*: \pi_N(j) \rightarrow \pi_N(Y)$ is, in fact, an isomorphism.)

Now if $\xi \in \pi_N(P_m, T)$ is represented by $(a, b): i \rightarrow k$, where $i: S^{N-1} \subseteq V^N$, then $h_*q_*(\xi)$ is represented by $hqb = fb: V^N, S^{N-1} \rightarrow Y, *$, and $(u, f)_*c_*(\xi)$ is represented by $(uca): i \rightarrow p$. But $uc = 0$, so that, making the necessary identification, $f_*c_*(\xi)$ is also represented by fb .

Let θ generate the cyclic infinite group $\pi_N(\bar{P}_m, \bar{T})$. Then

$$g_*(\eta) = c_*g_{T*}\partial\theta = \partial c_*g_{P*}\theta.$$

Since $\Omega f \simeq 0$ (indeed, f is a P -map), $\partial: \pi_N(j) \rightarrow \pi_{N-1}(F)$ is a monomorphism, so we must prove that $c_*g_{P*}\theta \neq 0$ and hence it is sufficient to prove that $h_*g_{Q*}\bar{q}_*\theta \neq 0$. But \bar{q}_* is an isomorphism and $\bar{q}_*\theta$ generates $\pi_N(\bar{Q})$. Thus the inequality $h_*g_{Q*}\bar{q}_*\theta \neq 0$ follows immediately from the hypothesis.

\dagger If such a map exists.

COROLLARY (3.4). *Under the same hypotheses F is not an H -space.*

Proof. If F were an H -space then every map $T \rightarrow F$ would be extendible to \bar{P}_m . But if g is extendible to \bar{P}_m then g_* certainly annihilates $\partial\pi_N(\bar{P}_m, T)$.

Remark (3.5). Lemma (3.3) and Corollary (3.4) in fact hold if $m = 2$.

We may apply the Corollary if X_i is $(n_i - 1)$ -connected, $\pi_{n_i}(X_i) = G_i$, $G_1 \otimes \dots \otimes G_m \neq 0$ and $h = 1$. We simply take $\alpha_1, \dots, \alpha_m$ to be elements such that $\alpha_1 \otimes \dots \otimes \alpha_m \neq 0$. Then $g_{Q*}\pi_N(\bar{Q})$ is the cyclic group generated by $\alpha_1 \otimes \dots \otimes \alpha_m$. We conclude in this case that

$$(3.6) \quad \Omega F \stackrel{H}{\simeq} \Omega P_m \times \Omega^2 Q, \quad F \text{ is not an } H\text{-space}.$$

We point out that if each X_i is an H -space then $P_m \times \Omega Q$ is an H -space and ΩF is H -equivalent to $\Omega(P_m \times \Omega Q)$, although F is not itself an H -space.

We could also apply the Corollary if $X_i = K(G_i, n_i)$ and $h: Q \rightarrow K(G, N)$ corresponds to a non-zero coefficient pairing $G_1 \otimes \dots \otimes G_m \rightarrow G$. We then conclude

$$(3.7) \quad \Omega F \stackrel{H}{\simeq} K(G_1, n_1 - 1) \times \dots \times K(G_m, n_m - 1) \times K(G, N - 2), \quad F \text{ is not an } H\text{-space}.$$

Clearly, higher Whitehead products should be so defined that one may say that, in Corollary (3.4), the fact that F is not an H -space, is recognized by the presence of a non-zero $(m - 1)$ st order Whitehead product $g_*(\eta)$; indeed, Porter's definition [9] permits this form of words. We may certainly deduce the existence of non-zero secondary Whitehead products (where there is no ambiguity as to definition) in the case $m = 3$.

Let us then take the case $m = 3$ of (3.6) and let us suppose further that $\text{nil } X_i = 1$, $i = 1, 2, 3$. Then it follows from (3.6) that $\text{nil } F = 1$ (although F is not an H -space). Thus all Whitehead products vanish in F so that secondary Whitehead products are defined for all triples of homotopy elements of F and the modulus of the operation vanishes. We show, in fact, that, if $j_*(\alpha_i) = \alpha_i$, then the secondary Whitehead product $[\alpha'_1, \alpha'_2, \alpha'_3] \in \pi_{N-1}(F)$ is non-zero, being precisely $g_*(\eta)$.

By a theorem of Stasheff [10], the condition $\text{nil } F = 1$ is precisely equivalent to the condition that, if $e: \Sigma\Omega F \rightarrow F$ is the natural map, then $\langle e, e \rangle: \Sigma\Omega F \vee \Sigma\Omega F \rightarrow F$ extends to $\Sigma\Omega F \times \Sigma\Omega F$. Now if F were an H -space we could take the union of an arbitrary number of summands $\Sigma\Omega F$ and extend $\langle e, \dots, e \rangle$ to their product. However, we prove, sharpening the statement that F is not an H -space,

THEOREM (3.8). *The map $\langle e, e, e \rangle: \Sigma\Omega F \vee \Sigma\Omega F \vee \Sigma\Omega F \rightarrow F$ cannot be extended to $\Sigma\Omega F \times \Sigma\Omega F \times \Sigma\Omega F$.*

Proof. We first identify $g_*(\eta)$ with $[\alpha'_1, \alpha'_2, \alpha'_3]$, as indicated above. To do this it is merely a matter of showing that $g|S^{n_i}$ represents α'_i . But

$$j_*g_* = j_*c_*g_{T*} = k_*g_{T*} = g_{P*}\bar{k}_*: \pi_{n_i}(\bar{T}) \rightarrow \pi_{n_i}(P_3).$$

Thus, if i_i is the class of the embedding of S^{n_i} in \bar{T} , then $j_*g_*i_i = \alpha_i$, so $g_*i_i = \alpha'_i$, as asserted. Now $g|S^{n_1} \vee S^{n_2} \vee S^{n_3}$ cannot be extended to $S^{n_1} \times S^{n_2} \times S^{n_3}$; for the obstruction to this extension is $[\alpha'_1, \alpha'_2, \alpha'_3]$ which is non-zero. Let $\bar{g}_i: S^{n_i} \rightarrow \Sigma\Omega F$ be a map such that $e\bar{g}_i =$

$g|S^{n_i}$; such a map is obtained by suspending the adjoint $S^{n_i-1} \rightarrow \Omega F$ of $g|S^{n_i}$. If $\langle e, e, e \rangle$ were extendible to a map $h: \Sigma \Omega F \times \Sigma \Omega F \times \Sigma \Omega F \rightarrow F$, then $h(\bar{g}_1 \times \bar{g}_2 \times \bar{g}_3)$ would be an extension of $g|S^{n_1} \vee S^{n_2} \vee S^{n_3}$ to $S^{n_1} \times S^{n_2} \times S^{n_3}$, contrary to the facts.

Remark (3.9). (a) Theorem (3.8) implies that, although $\text{nil } F = 1$, F is not only not an H -space but the map $e: \Sigma \Omega F \rightarrow F$ cannot be factored through an H -space.

(b) Note that the phenomenon expressed by Theorem (3.8) cannot appear in dual form. If e now stands for the embedding $e: A \rightarrow \Omega \Sigma A$ then $(e, e): A \rightarrow \Omega \Sigma A \times \Omega \Sigma A$ can be compressed into $\Omega \Sigma A \vee \Omega \Sigma A$ if and only if e factors through a suspension.

(c) It would be satisfactory to generalize Theorem (3.8) to the case of $m > 3$ factors. It seems likely that the methods of Porter [9] may make this possible.

We now elicit a further property of the spaces F we have been discussing in this section. Certainly if $\text{nil } F = 1$ and K is any space, then $\text{nil } F^K = 1$; however we will prove below that, although the spaces F are not themselves H -spaces, F^K is an H -space for a large class of (connected) spaces K . We suppose then that $f: X \rightarrow Y$ is a map which factors through some $q: P_m \rightarrow Q$; thus, if $m \geq 3$, f is a P -map. Assume now that K is a compact polyhedron and that its *weak category*[†] [2] satisfies the inequality (3.10)

$$(3.10) \quad w \text{ cat } K < m.$$

We prove

PROPOSITION (3.11). $f^K \simeq 0: X^K \rightarrow Y^K$.

Proof. It plainly suffices to prove that $q^K \simeq 0: P_m^K \rightarrow Q^K$. Now

$$P_m^K = (X_1 \times \dots \times X_m)^K = X_1^K \times \dots \times X_m^K,$$

since K is regular. Let us write $Q(K)$ for $Q(K, \dots, K)$ (m copies). Then there is clearly a transformation

$$s: X_1^K \times \dots \times X_m^K \rightarrow Q^{Q(K)},$$

given by

$$(3.12) \quad s(v_1, \dots, v_n) = v_1 \wedge \dots \wedge v_n, \quad v_i: K \rightarrow X_i;$$

and if, for any $u: A \rightarrow B$, C^u is the induced map $C^u: C^B \rightarrow C^A$, then

$$(3.13) \quad (Q^{qd})s = q^K: P_m^K \rightarrow Q^K.$$

Now certainly if $u \simeq u': A \rightarrow B$ then $C^u \simeq C^{u'}: C^B \rightarrow C^A$. Thus it follows from (3.13) that if $qd \simeq 0: K \rightarrow Q(K)$, then $q^K \simeq 0: P_m^K \rightarrow Q^K$, provided s is continuous.

Now s factors as

$$X_1^K \times \dots \times X_m^K \xrightarrow{s_1} (P_m, T)^{(P_m(K), T(K))} \xrightarrow{s_2} Q^{Q(K)},$$

where $P_m(K), T(K)$ have their obvious meanings, $s_1(v_1, \dots, v_n) = v_1 \times \dots \times v_n$, $v_i: K \rightarrow X_i$, and s_2 is given by passing to quotients; that is, $s_2(g) = \hat{g}$ if the diagram

[†] Renormalized so that $w \text{ cat } K = 0$ for contractible K ; thus $w \text{ cat } K < m$ if the map $K \xrightarrow{d} P_m(K) \xrightarrow{q} Q(K)$ is nullhomotopic.

$$\begin{array}{ccc} P_m(K) & \xrightarrow{g} & P_m \\ \downarrow q & \hat{g} & \downarrow q \\ Q(K) & \rightarrow & Q \end{array}$$

is commutative. Now let $[C, U]$ be a set of the subbase of neighbourhoods in $Q^{Q(K)}$, so that C is compact in $Q(K)$ and U is open in Q . Then

$$s_2^{-1}[C, U] = [q^{-1}C, q^{-1}U] \cap (P_m, T)^{(P_m(K), T(K))}.$$

Since K is compact Hausdorff $q^{-1}C$ is compact in $P_m(K)$, and $q^{-1}U$ is certainly open in P_m . Thus s_2 is continuous. To test the continuity of s_1 we may enlarge the range of s_1 and so regard s_1 as a map

$$s_1 : X_1^K \times \dots \times X_m^K \rightarrow P_m^{P_m(K)}.$$

Since $P_m(K)$ is regular, we may identify $P_m^{P_m(K)}$ with $X_1^{P_m(K)} \times \dots \times X_m^{P_m(K)}$ and then s_1 is given by

$$s_1 = X_1^{p_1} \times \dots \times X_m^{p_m} : X_1^K \times \dots \times X_m^K \rightarrow X_1^{P_m(K)} \times \dots \times X_m^{P_m(K)},$$

where $p_i : P_m(K) \rightarrow K$ is the projection onto the i th factor. Thus s_1 is continuous and so therefore is s . This completes the proof of Proposition (3.11).

COROLLARY (3.14). *Let $f : X \rightarrow Y$ factor through some $q : P_m \rightarrow Q$, $m \geq 2$, and let K be a compact polyhedron with $w \text{ cat } K < m$. Then if F is the fibre of f ,*

$$F^K \simeq (X \times \Omega Y)^K.$$

Proof. F^K is the fibre of $f^K : X^K \rightarrow Y^K$; but $f^K \simeq 0$ by Proposition (3.11). Thus in the sequence

$$(3.15) \quad (\Omega Y)^K = \Omega(Y^K) \xrightarrow{s^K} F^K \xrightarrow{j^K} X^K$$

there is a map $u : X^K \rightarrow F^K$ with $j^K u \simeq 1$. But (3.15) is then an induced (principal) fibration with cross-section so that F^K is a product.

$$F^K \simeq X^K \times (\Omega Y)^K = (X \times \Omega Y)^K.$$

We remark that if we take $K = S^1$, $m \geq 2$, then $w \text{ cat } K = 1 < m$ and we have just the familiar fact $\Omega F \simeq \Omega(X \times \Omega Y)$.

Summing up some of the facts proved in this section, we have the following

THEOREM (3.16). *Let X_i be an $(n_i - 1)$ -connected H -space, $n_i \geq 2$, $i = 1, \dots, m$, $m \geq 3$ and let $\pi_{n_1}(X_1) \otimes \dots \otimes \pi_{n_m}(X_m) \neq 0$. Then if F is the fibre of $q : P_m(X_1, \dots, X_m) \rightarrow Q(X_1, \dots, X_m)$.*

- (i) $\Omega F \simeq \Omega X_1 \times \dots \times \Omega X_m \times \Omega^2 Q$, so that $\text{nil } F = 1$;
- (ii) F is not an H -space;
- (iii) F^K is an H -space provided K is compact and $w \text{ cat } K < m$.

§IV. THE DUAL SITUATION

In this section we discuss the dual of the notion of the P -property for maps. We do not aim here at completeness, but merely sketch the dual procedure.

Let $f: Y \rightarrow X$ be a map in \mathcal{T} , let $j: X \rightarrow F$ be the cofibre of f and let $s: F \rightarrow \Sigma Y$ be the cofibre of j . There is thus an exact sequence

$$(4.1) \quad \dots \leftarrow \Sigma^2 Y \xleftarrow{\Sigma s} \Sigma F \xleftarrow{\Sigma j} \Sigma X \xleftarrow{\Sigma f} \Sigma Y \xleftarrow{s} F \xleftarrow{j} X \xleftarrow{f} Y.$$

We confine attention to those maps f for which $\Sigma f \simeq 0$; for such f

$$(4.2) \quad \Sigma F \simeq \Sigma X \vee \Sigma^2 Y,$$

and we may ask whether (4.2) may be given by an H' -equivalence (so that the comultiplication on ΣF is homotopy-equivalent to that on $\Sigma(X \vee \Sigma Y)$).

Using Lemma (2.3) again we infer the evident dual of Theorem (2.4) and use this dual to define and explore the concept of the P' -property. Precisely, then, we have

DEFINITION (4.3). *The map $f: Y \rightarrow X$ has property P' (or f is a P' -map) if $\Sigma j: \Sigma X \rightarrow \Sigma F$ has a left H' -homotopy-inverse.*

Then the dual of Theorem (2.4) yields three alternative characterizations of P' -maps, one of which shows that, for P' -maps, (4.2) is an H' -equivalence.

The analogues of (2.6)–(2.10) evidently hold, with obvious proofs. In the case of Proposition (2.8) one may point out that the dual reads as follows:

PROPOSITION (4.4). *Let $f: Y \rightarrow X$, $g: Z \rightarrow X$ and let $c: C \rightarrow Y \vee Z$ be the fibre of $\langle 0, 1 \rangle: Y \vee Z \rightarrow Z$. Then if f is a P' -map so is $\langle f, g \rangle \circ c: C \rightarrow X$.*

COROLLARY (4.5). *Let $q: Y \wr Z \rightarrow Y \vee Z$ be the fibre of the canonical map $Y \vee Z \rightarrow Y \times Z$. Then if f is a P' -map so is $\langle f, g \rangle \circ q$.*

We will be content to point out the analogue of Theorem (2.11) for the ordinary homotopy groups $\pi_n(X)$, leaving generalizations to the reader. Thus

THEOREM (4.6). (i) *The P' -classes in $\pi_n(X)$ form a subgroup. More generally if $\alpha_i \in \pi_{n_i}(X)$, $i = 1, \dots, m$, are P' -classes and T is an m -ary homotopy operation then $T(\alpha_1, \dots, \alpha_m)$ is a P' -class.*

(ii) *The Whitehead product of two homotopy classes is a P' -class if at least one of the factors is a P' -class. More generally, if $\alpha_i \in \pi_{n_i}(X)$, $i = 1, \dots, m$, and T is a normal m -ary homotopy operation then $T(\alpha_1, \dots, \alpha_m)$ is a P' -class if at least one of $\alpha_1, \dots, \alpha_m$ is a P' -class.*

(iii) *If $g: X_1 \rightarrow X_2$ then $g_*: \pi_*(X_1) \rightarrow \pi_*(X_2)$ maps P' -classes to P' -classes.*

The question of the existence of non-trivial P' -classes may be handled essentially as for P -classes. However we do not have a very convenient analogue of T_n in the category \mathcal{T} and it would probably be better to work in the category of c.s.s. groups. However, we may circumvent the difficulty in another way by observing that, in the light of Proposition (2.6) it is sufficient to demonstrate Theorem (2.14) for $n = 2$. Then T_2 does have a dual with which one can operate. Namely, we take the inverse limit of the fibre maps[†] $X_i \vee X_j \rightarrow X_i$

[†] That is, we replace the projection $X_i \vee X_j \rightarrow X_i$ by a fibre map before passing to the limit.

$i = 1, \dots, m, j = 1, \dots, m, i \neq j$. This then is a space T'_2 (a subspace of the product of the $X_i \vee X_j$), together with maps $T'_2 \rightarrow X_i \vee X_j$, and we get a factorization

$$(4.7) \quad W_m \xrightarrow{k'} T'_2 \rightarrow P_m$$

of the inclusion of the union $W_m = X_1 \vee \dots \vee X_m$ in the product P_m . Then (2.13) with $n = 2$, dualizes to the fibration sequence

$$(4.8) \quad Q'_2 \xrightarrow{q'} W_m \xrightarrow{k'} T'_2$$

and one may prove, dualizing the proof of Theorem (2.14).

THEOREM (4.9). *The map $q': Q'_2 \rightarrow W_m$ is a P' -map.*

It is not difficult to show that if $f: S^n \rightarrow X$ represents a 3-fold Whitehead product then f factors through $q': Q'_2(X, X, X) \rightarrow W_3(X, X, X)$ and so is a P' -map. Thus *all 3-fold Whitehead products are P' -classes*; indeed so are all 3-fold homotopy products in the sense of [6]. Thus if $f: Y \rightarrow X$ is an n -fold Whitehead product, $n \geq 3$, with cofibre F , then

$$(4.10) \quad \Sigma F \xrightarrow{H'} \Sigma X \vee \Sigma^2 Y,$$

and, provided ΣX is not contractible,

$$(4.11) \quad \text{conil } F = \text{conil } X;$$

for the definition of *conilpotency*, see [1].

However, even if X is an H' -space, F may not be so. The dual phenomenon was detected by the presence of higher Whitehead products; thus it is to be expected that here we obtain spaces F with non-vanishing higher cup products, i.e. Massey products. Indeed if $X = S^{n_1} \vee \dots \vee S^{n_m}$, $m \geq 3$, ι_i generates $\pi_{n_i}(S^{n_i})$, $N = n_1 + \dots + n_m$ and if $\alpha = [\iota_1, [\iota_2, [\dots \iota_m] \dots]]$, then $F = X \cup_{\alpha} e^{N-m+2}$ possesses non-zero Massey products [8]. On the other hand we have the proposition

PROPOSITION (4.12). *If A admits an H' -structure then all Massey products vanish in A .*

Proof. We are content to prove this explicitly for secondary products.† Let A be an H' -space. Then all cup-products vanish in A , so secondary products are universally defined and, of course, natural. Since A is a retract of a suspension it is thus sufficient to prove the assertion when A is a suspension. But if u, v, w are cohomology classes on the suspension complex A we may choose representative cocycles u', v', w' so that $u'v' = 0$, $v'w' = 0$. It is then plain that the Massey product $\langle u, v, w \rangle$ vanishes.

Thus we conclude that, for the special space F above,

$$(4.13) \quad \Sigma F \xrightarrow{H'} S^{n_1+1} \vee \dots \vee S^{n_m+1} \vee S^{N-m+3}, \text{conil } F = 1,$$

F is not an H' -space.

The phenomenon of Corollary (3.14) also has a dual analogue. We confine attention

† The general argument could proceed by induction.

to (generalized) Whitehead products. Let $\alpha_i \in \Pi(\Sigma A_i, X)$, $i = 1, \dots, m$, and let $W(\alpha_1, \dots, \alpha_m)$ be an m -fold Whitehead product of $\alpha_1, \dots, \alpha_m$ so that $W(\alpha_1, \dots, \alpha_m) \in \Pi(\Sigma(A_1 \wedge \dots \wedge A_m), X)$; we will also write $W(f_1, \dots, f_m)$ for the Whitehead map representing $W(\alpha_1, \dots, \alpha_m)$, where f_i represents α_i .

Set $B = A_1 \wedge \dots \wedge A_m$ and, for any compact polyhedron K , set $\tilde{A}_i = A_i \wedge K$. Then $\Sigma \tilde{A}_i$ may be identified with $\Sigma A_i \wedge K$ so that f_i determines $\tilde{f}_i = f_i \wedge 1: \Sigma \tilde{A}_i \rightarrow X \wedge K$. Also we may identify $\Sigma B \wedge Q(K)$ with $\Sigma(\tilde{A}_1 \wedge \dots \wedge \tilde{A}_m)$. We then prove easily

LEMMA (4.14). *The diagram*

$$\begin{array}{ccc} \Sigma B \wedge K & \xrightarrow{W(f_1, \dots, f_m) \wedge 1} & X \wedge K \\ \downarrow 1 \wedge qd & \nearrow W(\tilde{f}_1, \dots, \tilde{f}_m) & \\ \Sigma B \wedge Q(K) & & \end{array}$$

is commutative.

We infer

PROPOSITION (4.15). *If $\text{conil } K < m$, then $W(\alpha_1, \dots, \alpha_m) \wedge 1 = 0$.*

Proof. We may identify $1 \wedge qd: \Sigma B \wedge K \rightarrow \Sigma B \wedge Q(K)$ with $1 \wedge \Sigma(qd): B \wedge \Sigma K \rightarrow B \wedge \Sigma Q(K)$. Now $\Sigma(qd) \simeq 0$ if $\text{conil } K < m$ [5, Theorem (4.1)]. Thus Proposition (4.15) follows from Lemma (4.14).

COROLLARY (4.16). *If $f: Y \rightarrow X$ may be factored through an m -fold Whitehead product and if F is the cofibre of f , then*

$$F \wedge K \simeq (X \wedge K) \vee (\Sigma Y \wedge K),$$

for any compact polyhedron K with $\text{conil } K < m$.

Applying this to the special space F whose properties are described in (4.13), we may add to these properties the fact that $F \wedge K$ is an H' -space for any compact K with $\text{conil } K < m$.

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